Beyond the Thin Lens Approximation

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Abstract

We obtain analytic formulae for the null geodesics of Friedmann-Lemaître-Robertson-Walker spacetimes with scalar perturbations in the longitudinal gauge. We use these to provide a rigorous derivation of the cosmological lens equation. We obtain an expression for the magnification of a bundle of light rays in these spacetimes without restriction to static or thin lens scenarios. We show how the usual magnification matrix naturally emerges in the appropriate limits.

Keywords: Cosmology: Gravitational Lensing, Gravitation

1. Introduction

The bending of light by a single symmetric gravitational lens in a Euclidean space is shown in Fig. 1. The symmetry guarantees that the lines of sight from the observer, o, to the lens, l, and to both the lensed, and unlensed, image of the emitter, e, lie in the same plane, as do the angles α, β, and θ. $D(o,l)$ and $D(o,e)$ are the distances from the observer to the lens and emitter, respectively. We assume that the deflection angle, α, is small. Then, locally about the line of sight to the emitter's image, we may approximate the two-spheres at distances $D(o,l)$ and $D(o,e)$ from the observer as planes, called the lens plane and source plane respectively. $D(l,e)$ denotes the distance between these two planes. The assumption of small deflection angle also allows us to relate the lensing angles by $\beta = \theta - \alpha D(l,e)/D(o,e)$. This is the simplest example of the gravitational lens equation.

The generalization of this equation to more complicated lens structures and non-Euclidean background spaces proceeds by a number of steps. A general lens is not symmetric so that the angles α, β, and θ are not necessarily coplanar. To handle this, consider a set of cartesian axes with origin at the observer. Choose the x-axis to coincide with the line-of-sight to the image. Let α^i, where i runs over $\{2,3\}$, be the angle between the x-axis and the projection into the $x^i x$-plane of the line-of-sight vector from the deflection point, p, to the emitter. Similarly, let β^i be the angle between the projection into the $x^i x$-plane of the line-of-sight vector from the observer to the lens and the projection into the $x^i x$-plane of the line-of-sight vector from the observer to the unlensed image. Also, let θ^i be the angle between the x-axis and the projection into the $x^i x$-plane of the line-of-sight vector from the observer to the lens. To allow for non-Euclidean spatial geometries $D(o,e)$ is taken to be the angular-diameter distance in the background geometry from the

observer to the intersection of the x-axis with the source plane, and $D(l, e)$ is taken to be the angular-diameter distance in the background geometry between the deflection point, p, and the intersection of the x-axis with the source plane, p'. Then, again assuming small deflection angle,

$$\beta^i = \theta^i - \frac{D(l, e)}{D(o, e)}\alpha^i. \tag{1}$$

This is the standard cosmological gravitational lens equation (Schneider, Ehlers, and Falco 1993). The magnification matrix, $M^i{}_j = \partial\beta^i/\partial\theta^j$ contains information about the deformation of ray bundles connecting the observer and emitter. For example, the inverse of the determinant of this matrix is the magnification of an image relative to an unlensed image.

The purpose of the current paper is to address a number of subtle issues that arise when we attempt to justify mathematically the use of the lens equation (1) for calculations in our Universe, although most workers agree that the physical justifications for its use in observed lens systems are strong. First, there is the question of the best choice of distance factors. There exists a large literature addressing this question, primarily concerned with the appropriateness of the so-called Dyer-Roeder distances (Dyer and Roeder 1972, 1973; Ehlers and Schneider 1986; Futamase and Sasaki 1989; Watanabe and Tomita 1990; Watanabe, Sasaki, and Tomita 1992; Sasaki 1993). Our work suggests that within the framework of cosmological perturbation theory, the natural distance factors to use are those of the background. Hence, the choice of distance factors is equivalent to the choice of cosmological model, in agreement with the recent results of Sasaki (1993). The issue of which cosmological model is most appropriate must be addressed in its own right.

The second issue which should be addressed in any mathematical investigation into the lens equation concerns the accuracy of the approximation of an actual photon path by two geodesics of the background which join at a point near the lens, the deflection point, p. On physical grounds we expect this approximation to be good for systems for which the photon-lens interaction is localized: the thin-lens approximation. One purpose of our present work is to quantify the relationship between the actual path and that used in the lens equation.

Third, how are we to find the angles appearing in the lens equation from physical data? Generally, the α^i are taken to be those predicted by calculations in Einstein-de Sitter spacetime, since the overall curvature of space should not be important near p, where the light ray interacts with the lensing object. For static, thin lenses, these calculations write the deflection angle as a superposition of point mass deflection angles contributed by each mass element of the lens projected onto the lens plane (Schneider et al. 1993). For brevity we will term the resultant angle the "Einstein angle." Another purpose of this work is to derive this result from the full equations of light propagation under an appropriate set of mathematical approximations. The lens equation, for static, thin lenses, effectively assumes that the light path is described by the Jacobi equation of the background spacetime subject to an impulsive wavevector deflection at the lens plane by an angle equal to the

usual Einstein bending angle. We wish to quantify the level of approximation involved.

There have been two notable recent attempts to clarify the validity of the cosmological lens equation by deriving it from the optical scalar equations (Seitz, Schneider, and Ehlers 1994) and the Jacobi equation (Sasaki 1993). However, a crucial difference between these papers and the present work is that they treat the path of the light ray differently near to and far from the lens. It is precisely this assumption that we must eliminate if we hope to gain a more general lens equation able to quantify the errors implicit in equation (1).

Our approach is to investigate the cosmological lens equation as it emerges from the geodesic equation. We are able to do this by making use of a technique for constructing null geodesics in perturbed spacetimes (Pyne and Birkinshaw 1993). The conventions of the present paper are the same as those in that earlier work. The results of this letter come from applying this technique to FRW spacetimes with scalar perturbations in the longitudinal gauge. The calculations yielding the results presented here will appear in Pyne and Birkinshaw (1994). The results we report here are: analytic formulae (equations (4) and (5)) for light rays in the spacetime (2); a general expression for the magnification undergone by a bundle of light rays capable of handling non-static, geometrically thick, lenses (equations (10), (11), and (13)); and a demonstration that the usual deflection angle, lens equation (1) and magnification matrix are recovered in the appropriate approximations. To our knowledge, these are the first rigorous demonstrations of these results for perturbed FRW spacetimes.

2. The Deflection Angle

Our starting point is a choice for the metric of our Universe. We choose to work with FRW spacetimes with scalar perturbations written in the longitudinal gauge,

$$d\bar{s}^2 = a^2 \left[-(1+2\phi)d\eta^2 + (1-2\phi)\gamma^{-2}\left(dx^2+dy^2+dz^2\right)\right] \tag{2}$$

where $\gamma = 1+\kappa r^2/4$, κ being the spatial curvature parameter (± 1 or 0) and $r^2 = x^2+y^2+z^2$. Inhomogeneities are represented by the quasi-Newtonian potential, ϕ. For the order needed by us the expansion factor, a, is unperturbed from its Friedmann form (Jacobs, Linder, and Wagoner 1993). The metric (2) is also used by Seitz et al. (1994) and Sasaki (1993), so that direct comparison of results is possible, and recent work by Futamase (1989) and Jacobs et al. (1993) has shown that structure of galactic scales and greater in our Universe can be well modeled by metrics of this type. While results obtained with the metric (2) are not appropriate for lensing by gravitational waves or vector perturbations, it is possible to treat these cases by a similar method (see Pyne and Birkinshaw 1994).

Because the Friedmann expansion, a, plays the role of a conformal factor it is simplest to work with the null geodesics of ds^2, defined by $d\bar{s}^2 = a^2 ds^2$. Quantities in $d\bar{s}^2$ will always be written with an overbar. Light rays in these two metrics coincide and their (affine) parameterizations are related by $\bar{k}^\mu = a^{-2}k^\mu$. While our formulae are mostly written in terms of quantities in ds^2, our results apply to the actual spacetime, $d\bar{s}^2$. We note that by locating the observer at the spatial origin the radial null geodesics of $ds^{(0)2}$, (i.e. of that part of ds^2 independent of ϕ), obey $k^{(0)0} = 1$ and $k^{(0)i} = -\gamma e^i$, where e^i are

the direction cosines at the observer, so that $\sum_{i=1}^{3}(e^i)^2 = 1$ (McVittie 1964). The explicit solutions for the comoving radius and for γ along such rays are given by

$$
\begin{aligned}
r(\lambda) &= 2\tan_\kappa\left(\frac{\lambda_o - \lambda}{2}\right) \\
\gamma(\lambda) &= \sec^2_\kappa\left(\frac{\lambda_o - \lambda}{2}\right)
\end{aligned}
\tag{3}
$$

where λ_o is the affine parameter at the observer. The subscript κ on a trigonometric function denotes a set of three functions: for $\kappa = 1$ the trigonometric function itself, for $\kappa = -1$ the corresponding hyperbolic function, and for $\kappa = 0$ the first term in the series expansion of the function. The paths of the rays are $x^{(0)0} = \lambda$, $x^{(0)i} = re^i$.

The method presented in Pyne and Birkinshaw (1993) allows us to write the null geodesics of ds^2 as $x^\mu = x^{(0)\mu} + x^{(1)\mu}$. For light rays passing through the observer $x^{(0)\mu}$ is given as above and the separation vector is

$$
x^{(1)0}(\lambda) = +2\int_{\lambda_o}^{\lambda} du(u-\lambda)k^{(0)m}\phi_{,m}(u) \tag{4}
$$

and

$$
\begin{aligned}
x^{(1)i}(\lambda) = &-2k^{(0)i}\int_{\lambda_o}^{\lambda} du(u-\lambda)\frac{\partial\phi}{\partial\eta}(u) \\
&+ 2\gamma(\lambda)\int_{\lambda_o}^{\lambda} du\sin_\kappa(u-\lambda)\frac{1}{\gamma(u)}\left(\nabla_\perp\phi\right)^i
\end{aligned}
\tag{5}
$$

where ${\nabla_\perp}^i = g^{(0)mn}\left(\delta^i_n - e_n e^i\right)\partial_m$ is the transverse gradient operator. We see that the spatial separation is naturally written as a the sum of a longitudinal and a transverse term. The above integrals are taken over the background geodesic $x^{(0)\mu}$. Hence if these solutions are used for geometrically thick lenses, it will in general be necessary to apply an iterative procedure, incorporating a number of background paths. Such an approach is familiar from the usual multiple-lens plane theory (Schneider *et. al.* 1993) but we stress that formulae (4) and (5) allow the photon path be approximated to arbitrary accuracy, in contrast with the multiple lens plane theory where the continuum limit is not compatible with the asumptions underlying the theory. Precise statements of consistency will appear in Pyne and Birkinshaw (1994).

We can understand (4) and (5) by considering their relation to the equation of geodesic deviation. The Jacobi equation of $ds^{(0)2}$ subject to an arbitrary impulsive wavevector perturbation δk^μ at some affine parameter u, is solved by deviation vector δx^μ with spatial components

$$\delta x^i(\lambda) = -\frac{\gamma(\lambda)}{\gamma(u)} \sin_\kappa(u-\lambda)\delta k^i_\perp(u) - \frac{\gamma(\lambda)}{\gamma(u)}(u-\lambda)\delta k^i_\parallel(u) \tag{6}$$

where $\delta k^i_\perp = \left(\delta^i_j - e^i e_j\right)\delta k^j$ is the impulse in the transverse direction, and $\delta k^i_\parallel = \delta k^i - \delta k^i_\perp$ is the longitudinal impulse.

Comparing (5) with (6), produces the following interpretation of the spatial components of our solution for the separation, $x^{(1)i}$: the photon path differs from a background path because of a continuous sequence of impulsive perturbations

$$\delta k^i = -2\left(\nabla_\perp \phi\right)^i + 2k^{(0)i}\frac{\partial \phi}{\partial \eta}. \tag{7}$$

In fact, the form of the impulse can be gained directly from our equations by differentiating the spatial separations with respect to the affine parameter, and inserting a delta function at λ_l into the integrand which forces the integrand to vanish except at the lens plane. This gives

$$k^{(1)i}\left(\lambda_l\right) = -2\left(\nabla_\perp \phi\right)^i\left(\lambda_l\right) + 2k^{(0)i}\left(\lambda_l\right)\frac{\partial \phi}{\partial \eta}\left(\lambda_l\right), \tag{8}$$

which is exactly the impulse found above, equation (7).

At this point we need only do a little work to recover the Einstein angle from our equations. Establish a set of cartesian axes at the observer and choose the unperturbed wavevector $k^{(0)} = (1, -\gamma, 0, 0)$. Consider a static, localized perturbation in the xy-plane. Then the angle represented by the impulse perturbation found above, that is, the angle $k^{(0)i}\left(\lambda_l\right) + k^{(1)i}\left(\lambda_l\right)$ makes with $k^{(0)i}\left(\lambda_l\right)$, is given by

$$\hat{\alpha}^y = \frac{-2\left(\nabla_\perp \phi\right)^y}{\gamma\left(\lambda_l\right)} = -2\gamma\left(\lambda_l\right)\phi_{,y}. \tag{9}$$

The factor of $\gamma\left(\lambda_l\right)$ is present only because our co-ordinates are scaled in an unusual way at the lens plane. If we make the co-ordinate change $x^{i\prime} = x^i/\gamma\left(\lambda_l\right)$ the metric on the lens plane becomes Minkowskian and, locally near the deflector, y' and z' serve as normal co-ordinates on the lens plane. In these co-ordinates $\phi_{,y'}$ on the lens plane takes on the usual Newtonian form (with origin shifted from the lens, accounting for the unusual minus sign). Since our lensing angle $-2\gamma\left(\lambda_l\right)\phi_{,y} = -2\phi_{,y'}$ the integrated impulse lensing angle for a localized perturbation is exactly the Einstein deflection angle. We emphasize that this is the first time this has been shown rigorously for the curved FRW spacetimes.

For completeness we note that the timelike component of the separation can also be analyzed by comparison to the Jacobi equation. The Jacobi equation of $ds^{(0)2}$ for an impulse wavevector perturbation δk^μ at affine parameter u results in a timelike component

of the deviation vector $\delta x^0 = -(u-\lambda)\delta k^0(u)$. Comparison with our solution for the separation reveals that the time delay undergone by the light ray relative to the fiducial background ray may be considered to result from a sequence of impulses $\delta k^0 = -2k^{(0)m}\phi_{,m}$.

3. The Magnification

We now want to examine the magnification undergone by a bundle of light rays. We define this after Schneider et al. (1993) in the following way. Suppose a source of given physical size at some redshift is observed to subtend solid angle $d\Omega$. An identical source observed at identical redshift placed in an FRW spacetime would subtend solid angle $d\Omega^{(0)}$. The magnification M is defined to be $d\Omega/d\Omega^{(0)}$.

We can construct a bundle of light rays in the spacetime (2) which emanate from a source and converge at an observer by varying the direction cosines, e^i, in equations (4) and (5). This yields $d\Omega$ in terms of the redshift and proper size of the source. A similar relationship is easy to derive for a source in the background spacetime. Comparison of the two yields M. If the four-velocities of the observer and emitter are written $u^{\mu}_{o(e)} = (1-\phi_{o(e)}, v^i_{o(e)})$, which are correct to first order in ϕ and v, then

$$M = \left(1 + 2\phi_o + 2\left[v^i k_i^{(0)}\right]_o^e - 2k_e^{(1)0} + 2\cot_\kappa\left(\lambda_o - \lambda_e\right)\delta\lambda_e + \kappa\sin_\kappa\left(\lambda_o-\lambda_e\right)e_i x_e^{(1)i}\right) \Big/ \left(\mathrm{Det}M^i{}_j\right) \tag{10}$$

where $[f]_o^e = f(\lambda_e) - f(\lambda_o)$ and the subscript e indicates evaluation at λ_e. Here $\delta\lambda_e$ enforces the equal redshift aspect of the definition. It is set by $1+z(\lambda_e+\delta\lambda_e) = 1+z^{(0)}(\lambda_e)$ and is given by

$$\delta\lambda_e = \frac{a_e}{\dot{a}_e}\left[\phi - v^i k_i^{(0)} - \dot{a}a^{-1}x^{(1)0} + k^{(1)0}\right]_o^e. \tag{11}$$

$M^i{}_j$ is the magnification matrix, of dimension 2×2,

$$M^i{}_j = I_2 + \frac{1}{r}\frac{\partial x^{(1)i}}{\partial e^j}, \tag{12}$$

where I_2 is the two-dimensional identity matrix and the indices i, j run only over the transverse spatial dimensions. For simplicity, we will henceforth assume the unperturbed wavevector $k^{(0)} = (-1, \gamma, 0, 0)$ so that i, j run over $\{2, 3\}$. Keeping in mind the transverse nature of these indices we can write $\partial/\partial e^j = \partial/\partial\theta^j$, with θ the vectorial angle of equation (1).

Explicit calculation yields

$$
\begin{aligned}
M^i{}_j = \delta^i_j &+ \frac{2}{\sin_\kappa(\lambda_o - \lambda_e)}\delta^i_j \int_{\lambda_o}^{\lambda_e} du\ (u-\lambda_e)\,\frac{\partial\phi}{\partial\eta}(u) \\
&- \frac{2}{\sin_\kappa(\lambda_o - \lambda_e)} \int_{\lambda_o}^{\lambda_e} du\ \sin_\kappa(u-\lambda_e)\,\gamma(u) \\
&\times \left[\delta^i_j \frac{\partial\phi}{\partial x}(u) - r(u)\gamma^{-2}(u) g^{(0)ik}(u)\phi_{,kj}(u)\right].
\end{aligned} \tag{13}
$$

To see how the usual magnification matrix is contained in equation (13) we rewrite the second integral using the angular diameter distance of $d\bar{s}^{(0)}$, $\bar{D}(u,\lambda) = a(\lambda)\sin_\kappa(u-\lambda)$. In this way

$$
\begin{aligned}
M^i{}_j = \delta^i_j &+ \frac{2}{\sin_\kappa(\lambda_o - \lambda_e)}\delta^i_j \int_{\lambda_o}^{\lambda_e} du\ (u-\lambda_e)\,\frac{\partial\phi}{\partial\eta}(u) \\
&- \frac{1}{\bar{D}(\lambda_o,\lambda_e)} \int_{\lambda_o}^{\lambda_e} du\, \bar{D}(u,\lambda_e)\,\frac{\partial\hat{\alpha}^i}{\partial\theta^j},
\end{aligned} \tag{14}
$$

with the two-dimensional projected angle

$$
\hat{\alpha}^i = -2\frac{(\nabla_\perp\phi)^i}{\gamma(u)}. \tag{15}
$$

We now consider lensing by a static, geometrically thin lens. The first term in equation (14) vanishes and we approximate the angular diameter factor as constant over the region for which the potential is important so that

$$
M^i{}_j \approx \delta^i_j - \frac{\bar{D}(\lambda_l, e)}{\bar{D}(o,e)}\frac{\partial}{\partial\theta^j}\int_{\lambda_o}^{\lambda_e} du\,\hat{\alpha}^i \tag{16}
$$

We have already seen that the integral of $\hat{\alpha}^i$ over the background path produces the Einstein deflection angle. As a result we conclude that our equation has reproduced the usual magnification matrix defined by the θ-gradient of equation (1).

4. **Summary**

We have presented formulae (4), (5) for the null geodesics intersecting an observer's worldline in an important class of perturbed spacetimes, FRW backgrounds with scalar perturbations, in the longitudinal gauge. We have used these equations to obtain a general formula (10) for the magnification of ray bundles in these spacetimes. With this, we can show for the first time how the usual lens equation (1) and magnification matrix are recovered in these spacetimes without dividing light paths into near and far lens regions. In forthcoming papers we will consider the implications of these results for practical lensing calculations.

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Figure Captions

Figure 1. Gravitational lensing by a single symmetric lens.